

Finding Optimal Minors of Valuated Bimatroids

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Abstract—As a variant of “valuated matroid” of Dress and Wenzel, we define the concept of a “valuated bimatroid” to investigate the combinatorial properties of the degree of subdeterminants of a rational function matrix. Two algorithms are developed for computing the maximum degree of a minor of specified order; the algorithms are valid also for “valuated bimatroids” in general.

Keywords—Valuated matroid, Concavity, Degree of subdeterminant, Smith-McMillan form of a rational function matrix.

1. INTRODUCTION

Let $A(x) = (A_{ij}(x))$ be an $m \times n$ matrix with $A_{ij}(x)$ being a rational function in x with coefficients from a field F , and denote by δ_k the highest degree of a minor of order k of $A(x)$. That is,

$$\delta_k = \max\{w(I, J) \mid |I| = |J| = k\} \quad (1)$$

with

$$w(I, J) = \deg \det A[I, J], \quad (2)$$

where $A[I, J]$ denotes the submatrix of A with row-set I and column-set J , and the degree of a rational function $f(x) = p(x)/q(x)$ (with $p(x), q(x) \in F[x]$) is defined by $\deg f = \deg p - \deg q$. We assume $\det A[\emptyset, \emptyset] = 1$ and $w(\emptyset, \emptyset) = 0$.

The present paper develops two algorithms for computing δ_k by exploiting the combinatorial properties of the function $w(I, J)$. To be specific, we first define the concept of a “valuated bimatroid” as a variant of “valuated matroid” introduced by Dress and Wenzel [2,3]; a valuated bimatroid is nothing but a valuated matroid in a disguise, just as a bimatroid [4] (or a linking system [5]) can be identified with a matroid with a fixed base. Then, we derive two theorems; the first (Theorem 1) stating the concavity of δ_k as a function of k , and the second (Theorem 2) revealing the nesting structure of the maximizers (I, J) of $w(I, J)$. Finally we develop, based on those theorems, two different algorithms for computing δ_k that work for valuated bimatroids in general, and we mention their applications to rational function matrices.

The results of this paper not only contribute to the theory of valuated matroid but also have fundamental engineering significance. For example, when δ_k is defined for a rational function matrix $A(x)$ as above, we can obtain the Smith-McMillan form at infinity (known also as the structure at infinity in the literature of control theory [6]) of $A(x)$ by computing δ_k ($k = 1, 2, \dots$).

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When $A(x)$ is a regular pencil, on the other hand, the δ_k ($k = 1, 2, \dots$) determine the structural indices of its Kronecker form [7] (and hence the index of nilpotency) which plays an important role in the analysis of differential-algebraic equations (DAEs). See [1] for details.

2. VALUATED BIMATROID

We introduce the concept of a “valuated bimatroid” as a variant of valuated matroids as follows. Let R and C be disjoint finite sets and let w be a map from $2^R \times 2^C$ into $\mathbf{R} \cup \{-\infty\}$. With w , we associate the map $v : 2^{R \cup C} \rightarrow \mathbf{R} \cup \{-\infty\}$, defined by

$$v(I \cup J) := w(R - I, J), \quad I \subseteq R, \quad J \subseteq C. \quad (3)$$

We define (R, C, w) to be a valuated bimatroid if $(R \cup C, v)$ is a valuated matroid (as defined in [2,3]) with $v(R) \neq -\infty$. This means in particular that

$$w(\emptyset, \emptyset) \neq -\infty \quad (4)$$

and

$$w(I, J) = -\infty, \quad \text{for } (I, J) \notin S, \quad (5)$$

where

$$S = \{(I, J) \mid |I| = |J|, I \subseteq R, J \subseteq C\}.$$

By translating the exchange axiom for valuated matroids v into conditions on w , we see that (R, C, w) is a valuated bimatroid if and only if w satisfies (4), (5), and the exchange properties (E1) and (E2) below for any $(I, J) \in S$ and $(I', J') \in S$:

(E1) For any $i' \in I' - I$, at least one of the following two assertions holds:

- (a1) $\exists j' \in J' - J : w(I, J) + w(I', J') \leq w(I + i', J + j') + w(I' - i', J' - j')$,
- (b1) $\exists i \in I - I' : w(I, J) + w(I', J') \leq w(I - i + i', J) + w(I' - i' + i, J')$.

(E2) For any $j \in J - J'$, at least one of the following two assertions holds:

- (a2) $\exists i \in I - I' : w(I, J) + w(I', J') \leq w(I - i, J - j) + w(I' + i, J' + j)$,
- (b2) $\exists j' \in J' - J : w(I, J) + w(I', J') \leq w(I, J - j + j') + w(I', J' - j' + j)$.

We define the rank r of (R, C, w) by

$$r = \max\{|I| \mid \exists (I, J) \in S : w(I, J) \neq -\infty\}.$$

REMARK 1. (E1) and (E2) together can be put in a more compact form as

(E12) For any $X' \subseteq I' - I$ and $Y \subseteq J - J'$ with $|X'| + |Y| = 1$ there exist $X \subseteq I - I'$ and $Y' \subseteq J' - J$ with $|X| + |Y'| = 1$ such that

$$w(I, J) + w(I', J') \leq w(I - X + X', J - Y + Y') + w(I' - X' + X, J' - Y' + Y).$$

REMARK 2. The Grassmann-Plücker relations (or the Laplace expansions) imply that $w(I, J) := \deg \det A[I, J]$ defines a valuated bimatroid (R, C, w) with $w(\emptyset, \emptyset) = 0$. As is obvious from [3], a valuated bimatroid can be defined more generally from any matrix over a field with nonarchimedean valuation.

REMARK 3. The family $\{(I, J) \mid w(I, J) \neq -\infty\}$ constitutes the family of linked pairs of a bimatroid (or a linking system) in the sense of [5]. This follows from the correspondence between matroids and bimatroids as well as the fact [3, Section 1, Remark (i)] that the family of bases of a valuated matroid constitutes the family of bases of a matroid.

To state our results we introduce notations:

$$\begin{aligned} S_k &= \{(I, J) \mid |I| = |J| = k, I \subseteq R, J \subseteq C\}, \\ \delta_k &= \max\{w(I, J) \mid (I, J) \in S_k\}, \quad 0 \leq k \leq r, \\ M_k &= \{(I, J) \in S_k \mid w(I, J) = \delta_k\}, \quad 0 \leq k \leq r. \end{aligned}$$

THEOREM 1. *The following inequality holds*

$$\delta_{k-1} + \delta_{k+1} \leq 2\delta_k, \quad \text{for } 1 \leq k \leq r-1. \quad (6)$$

PROOF. Let $(I_-, J_-) \in M_{k-1}$, $(I_+, J_+) \in M_{k+1}$ and $i \in I_+ - I_-$. From (E1) it follows that

$$\exists j \in J_+ - J_- : \delta_{k-1} + \delta_{k+1} \leq w(I_- + i, J_- + j) + w(I_+ - i, J_+ - j)$$

or

$$\exists i' \in I_- - I_+ : \delta_{k-1} + \delta_{k+1} \leq w(I_- - i' + i, J_-) + w(I_+ - i + i', J_+).$$

In the first case, the right-hand side is bounded by $2\delta_k$ since $(I_- + i, J_- + j) \in S_k$ and $(I_+ - i, J_+ - j) \in S_k$; thus (6) is established. In the second case, the right-hand side is bounded by $\delta_{k-1} + \delta_{k+1}$, and therefore $(I_- - i' + i, J_-) \in M_{k-1}$ and $(I_+ - i + i', J_+) \in M_{k+1}$. Putting $(\tilde{I}_-, \tilde{J}_-) = (I_-, J_-)$, $(\tilde{I}_+, \tilde{J}_+) = (I_+ - i + i', J_+)$, we have $(\tilde{I}_-, \tilde{J}_-) \in M_{k-1}$, $(\tilde{I}_+, \tilde{J}_+) \in M_{k+1}$ and $|\tilde{I}_- - \tilde{I}_+| = |I_- - I_+| - 1$.

By applying the above argument repeatedly, we will end up with the first case since $|I_- - I_+|$ decreases while the second case applies. ■

Next we look at the families M_k ($k = 0, 1, \dots, r$) of the maximizers of w .

LEMMA 1. *Let $(I_k, J_k) \in M_k$ and $(I_-, J_-) \in M_{k-1}$, where $1 \leq k \leq r$.*

(1) *For any $i \in I_k - I_-$, either (a'1) or (b'1) (or both) holds, where*

$$\begin{aligned} \text{(a'1)} \quad & \exists j \in J_k - J_- : (I_- + i, J_- + j) \in M_k, (I_k - i, J_k - j) \in M_{k-1}, \\ \text{(b'1)} \quad & \exists i' \in I_- - I_k : (I_- - i' + i, J_-) \in M_{k-1}, (I_k - i + i', J_k) \in M_k. \end{aligned}$$

(2) *For any $j \in J_k - J_-$, either (a'2) or (b'2) (or both) holds, where*

$$\begin{aligned} \text{(a'2)} \quad & \exists i \in I_k - I_- : (I_k - i, J_k - j) \in M_{k-1}, (I_- + i, J_- + j) \in M_k, \\ \text{(b'2)} \quad & \exists j' \in J_- - J_k : (I_k, J_k - j + j') \in M_k, (I_-, J_- - j' + j) \in M_{k-1}. \end{aligned}$$

PROOF. This follows from inequalities like those in the proof of Theorem 1. ■

THEOREM 2. *For any $(I_k, J_k) \in M_k$ with $1 \leq k \leq r-1$, there exist $(I_l, J_l) \in M_l$ ($0 \leq l \leq r$, $l \neq k$) such that $(\emptyset =) I_0 \subseteq I_1 \subseteq \dots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \dots \subseteq I_r$ and $(\emptyset =) J_0 \subseteq J_1 \subseteq \dots \subseteq J_{k-1} \subseteq J_k \subseteq J_{k+1} \subseteq \dots \subseteq J_r$.*

PROOF. We show the existence of such (I_{k-1}, J_{k-1}) . The existence of (I_{k+1}, J_{k+1}) can be shown in a similar manner, and the existence of the other (I_l, J_l) (with $|l - k| \geq 2$) follows by induction.

Let $(I_-, J_-) \in M_{k-1}$. If $I_- \not\subseteq I_k$, we apply Lemma 1 (1) to obtain (a'1) or (b'1). In case of (a'1), we are done with $(I_{k-1}, J_{k-1}) = (I_k - i, J_k - j)$. In case of (b'1), we put $\tilde{I}_- = I_- - i' + i$, $\tilde{J}_- = J_-$ to get $(\tilde{I}_-, \tilde{J}_-) \in M_{k-1}$ with $|\tilde{I}_- - I_k| = |I_- - I_k| - 1$.

Applying the above argument repeatedly, we will arrive at the case (a'1) or otherwise obtain $(I_-, J_-) \in M_{k-1}$ with $I_- \subseteq I_k$. Note that J_- remains unchanged in the latter case. Then, we apply Lemma 1 (2) repeatedly in a similar way as long as $J_- \not\subseteq J_k$, eventually arriving at $(I_-, J_-) \in M_{k-1}$ with $I_- \subseteq I_k$ and $J_- \subseteq J_k$. ■

3. ALGORITHMS

Obviously, Theorem 2 suggests the following incremental (or greedy) algorithm for computing δ_k for $k = 0, 1, \dots, r$. This algorithm involves $O(r|R||C|)$ evaluations of w to compute the whole sequence $(\delta_0, \delta_1, \dots, \delta_r)$.

Algorithm I

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 $I_0 := \emptyset; J_0 := \emptyset;$ 
for  $k := 1, 2, \dots$  do
  Find  $i \in R - I_{k-1}, j \in C - J_{k-1}$  that maximizes  $w(I_{k-1} + i, J_{k-1} + j)$ 
  and put  $I_k := I_{k-1} + i, J_k := J_{k-1} + j$  and  $\delta_k := w(I_k, J_k)$ .

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The iteration stops when $w(I_k, J_k) = -\infty$, and then the rank r is given by $k - 1$.

The second algorithm is based on Theorem 1 and relies on the greedy algorithm defined in [2] for valuated matroids. The concavity of δ_k implies that for α satisfying

$$\delta_k - \delta_{k-1} \geq \alpha \geq \delta_{k+1} - \delta_k, \quad (7)$$

the maximum of $\delta_l - \alpha l$ over $0 \leq l \leq r$ is attained by $l = k$. From this, it follows that

$$\delta_k = k\alpha + \max\{w_\alpha(I, J) \mid (I, J) \in S\}, \quad (8)$$

where

$$w_\alpha(I, J) = w(I, J) - \alpha|I|,$$

since

$$\max_{(I, J) \in S} w_\alpha(I, J) = \max_l \max_{(I, J) \in S_l} w_\alpha(I, J) = \max_l (\delta_l - \alpha l) = \delta_k - \alpha k.$$

It should be noted here that (R, C, w_α) is again a valuated bimatroid and, hence, $\max_{(I, J) \in S} w_\alpha(I, J)$ can be computed efficiently by applying the greedy algorithm of [2] to the valuated matroid associated with w_α as indicated in (3).

We can find

$$\begin{aligned}
k_+(\alpha) &\equiv \max\{k \mid \max_l (\delta_l - \alpha l) = \delta_k - \alpha k\}, \\
k_-(\alpha) &\equiv \min\{k \mid \max_l (\delta_l - \alpha l) = \delta_k - \alpha k\}
\end{aligned}$$

by maximizing (resp. minimizing) $|I|$ among the maximizers of $w_\alpha(I, J)$ by means of a slightly modified version of the greedy algorithm of [2].

The following algorithm computes the rank r and δ_k ($k = 0, 1, \dots, r$) by searching for appropriate values of α . It requires $O(r|R||C|)$ evaluations of w .

Algorithm II

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 $\delta_0 := w(\emptyset, \emptyset);$ 
Let  $\alpha$  be sufficiently small ( $\alpha < 0, |\alpha|$ : large);
Maximize  $w_\alpha$  to find  $r = k_+(\alpha) = k_-(\alpha)$  and  $\delta_r$ ;
if  $r \geq 2$  then search(0,  $r$ ).

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Here, the procedure “search(k_1, k_2)” is defined when $k_1 + 2 \leq k_2$ as follows.

procedure search(k_1, k_2)

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 $\alpha := (\delta_{k_2} - \delta_{k_1}) / (k_2 - k_1);$ 
Maximize  $w_\alpha$  to find  $k_+ = k_+(\alpha), k_- = k_-(\alpha), \delta_+ = \delta_{k_+}$  and  $\delta_- = \delta_{k_-}$ ;
for  $k_- < k < k_+$  do  $\delta_k := ((k - k_-)\delta_+ + (k_+ - k)\delta_-) / (k_+ - k_-)$ ;
if  $k_1 + 2 \leq k_-$  then search( $k_1, k_-$ );
if  $k_+ + 2 \leq k_2$  then search( $k_+, k_2$ ).

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REMARK 4. The above algorithms, when applied to w of (2), yield new algorithms for computing the Smith-McMillan form at infinity (see [1, 8] for other algorithms). In this case, $w(I, J)$ can be evaluated either by interpolation (see [9] for rational function interpolation) or by combinatorial relaxation [10]. The former is superior in theoretical complexity whereas the latter in practical efficiency.

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